Factoring – The Greatest Common Factor

❑ *Introduction*

Do you remember the concept of a **prime number**? Here's a hint: 13 is prime, but 15 is not prime.

A number is ***prime*** if it has exactly two factors.

For example, 13 is prime because 13 has exactly two factors (divisors), just 1 and 13. The number 2 is also prime, because its only factors are 1 and 2. But 15 is not prime — this is because 15 has more than two factors; in fact, it has four factors: 1, 3, 5, and 15. The number 1 is also not prime, since it does not have two factors; its only factor is 1. Thus, the first few primes are

> The number *a* is a ***factor*** of *b* if *a* divides into *b* evenly (without remainder). For example, 10 is a factor of 30, but 8 is not a factor of 30.

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, and so on, . . . ***forever***!

Believe it or not, every number (bigger than 3) that isn't prime can be written as a product of primes. For example, 90 is not prime, but 90 can be written as a product of primes:

$$90 = 2 \times 3 \times 3 \times 5$$

which we can write as

$$90 = 2 \times 3^2 \times 5$$

> The largest known prime number (as of Oct, 2024) is
>
> $$2^{136,279,841} - 1$$
>
> and contains over 41 million digits.
>
> If you wrote this prime number down, writing one digit per second, 24/7, it would take you well over a year, filling thousands of pages.

Notice that all the factors on the right side of the equality are primes, and that their product is surely 90.

This product of primes is called the ***prime factorization*** of 90. Writing 90 as $2 \times 3 \times 15$ is not its prime factorization because 15 is not prime. Basically, then,

Factoring *is the art of expressing something as a product of things — things which cannot be broken down any further.*

Homework

1.
   a. How many primes are there?
   b. What is the smallest prime?
   c. What is the only even prime?
   d. How many primes are there less than 100?
   e. Is the number $123 \times 56{,}893$ prime? Prove your answer.

2.
   a. Find the prime factorization of 306.
   b. Find the prime factorization of 1,024.

❑ A DIFFERENT VIEW OF THE DISTRIBUTIVE PROPERTY

We've generally viewed the distributive property in a form like this:

$$A(B + C) = AB + AC \qquad \text{DISTRIBUTING}$$

and we saw the power of such a law in simplifying expressions and solving complicated equations. But the distributive property is a statement of equality — we might find it useful to flip it around the equals sign and write it as

$$AB + AC = A(B + C) \qquad \text{FACTORING}$$

This provides a whole new perspective. It allows us to take a pair of terms, the <u>sum</u> $AB + AC$, find the **common factor** A (it's in both terms), and "pull" the A out in front, and write the <u>sum</u> $AB + AC$ as the <u>product</u> $A(B + C)$.

This use of the distributive property in reverse is called ***factoring***. Notice that using the distributive property in reverse converts the two terms $AB + AC$, into one term, $A(B + C)$.

For example, suppose we want to factor $9a + 9b$; that is, we want to convert $9a + 9b$ from a sum to a product. We notice that 9 is a common factor of the two terms. We pull the 9 away from both terms, and put it out in front to get $9(a + b)$, and we're done factoring:

$9a + 9b$ factors into $9(a + b)$.

To check this answer, distribute $9(a + b)$ and you'll get the original $9a + 9b$.

RECAP:

$9(a + b)$	Distribute →	$9a + 9b$	One term to two terms
$9a + 9b$	Factor →	$9(a + b)$	Two terms to one term

❑ *FACTORING*

<u>EXAMPLE 1:</u> **Factor each expression:**

A. $7x + 7y = 7(x + y)$

B. $3x + 12 = 3(x + 4)$

C. $ax + bx = x(a + b)$

D. $Rw - Ew = w(R - E)$

E. $9z + 9 = 9(z + 1)$

F. $mn - m = m(n - 1)$

G. $-6R + 8 = -2(3R - 4)$

Alternatively, we could pull out a positive 2, yielding $2(-3R + 4)$.

H. $-ax - at = -a(x + t)$

I. $-x + 5 = -(x - 5)$

J. $-n - 9 = -(n + 9)$

K. $6r + 8s - 10t = 2(3r + 4s - 5t)$

Note that every problem in the preceding example can be checked by distributing the answer. Our next example shows how we can factor out a variable in a quadratic expression.

<u>EXAMPLE 2:</u> **Factor each expression:**

A. $x^2 + 3x = x(x + 3)$

B. $n^2 - 7n = n(n - 7)$

C. $t^2 + t = t(t + 1)$

D. $y^2 - y = y(y - 1)$

E. $m^2 - 10m = m(m - 10)$

F. $a^2 + 40a = a(a + 40)$

Sometimes we can pull out a number and a variable.

<u>EXAMPLE 3:</u> **Factor: $2a^2 - 8a$**

<u>Solution:</u> What common factor can be pulled out in front? Since 2 is a factor of both terms, it can be pulled out. But a is a common factor, so it needs to come out in front, also. In other words, the quantity $2a$ is common to both terms (and it's the <u>largest</u> quantity that is <u>common</u> to both terms). So we factor it out and leave inside the parentheses what must be left. Thus, $2a^2 - 8a$ factors into

$$\boxed{2a(a - 4)}$$

Check by distributing

EXAMPLE 4: **Factor: $20n + 50$**

Solution: What factor is common to both terms that can then be pulled out in front? We have a little dilemma here. There are three numbers we could factor out: 2, 5, and 10. Let's agree to pull out the 10, since it's the largest factor that is common to both terms. Therefore, $2n + 50$ factors into

$$\boxed{10(2n + 5)}$$

New Terminology: In Example 3 we factored out the quantity $2a$ because it was common to both terms, and it was the greatest common factor. In Example 4 we factored out the number 10 because it was the greatest factor that was common to both terms. Each quantity, the $2a$ and the 10, is called the ***greatest common factor***, or **GCF**.

Homework

3. How would you convince your buddy that factoring $20x + 30y$ produces a result of $10(2x + 3y)$?

4. Your stubborn friend believes that $6w + 9z$ factors to $6(w + 3z)$. Prove her wrong.

5. Finish the factorization of each expression:

   a. $wx + wz = w(\quad)$ b. $4P - 4Q = 4(\quad)$

   c. $9x - 36 = 9(\quad)$ d. $8y - 12t = 4(\quad)$

   e. $7u + 7 = 7(\quad)$ f. $-2n + 8 = -2(\quad)$

   g. $-a + b = -(\quad)$ h. $-c - d = -(\quad)$

   i. $2x + 4y - 8z = 2(\quad)$ j. $aw - au + az = a(\quad)$

k. $14x^2 - 21x = 7x(\quad)$ l. $20a^2 + 30a - 40 = 10(\quad)$

6. Factor each expression:

a. $3P + 3Q$ b. $9n - 27$ c. $cn + dn$

d. $wx - xy$ e. $7t - 7$ f. $x + xy$

g. $-8L + 10$ h. $-ab - bc$ i. $-u - 5$

j. $-z - x + 10$ k. $2x + 2y + 2z$ l. $5a - 10b + 15c$

7. Finish the factorization of each expression:

a. $4a + 8b = 4(\quad)$ b. $9u^2 - 3u = 3u(\quad)$

c. $15Q - 45R = 15(\quad)$ d. $18x^2 + 12x = 6x(\quad)$

e. $10y^2 - 20y = 10y(\quad)$ f. $50a + 75b = 25(\quad)$

g. $7t^2 + 28t = 7t(\quad)$ h. $48w - 64z = 16(\quad)$

i. $100a^2 - 80a = 20a(\quad)$ j. $47y^2 + 47y = 47y(\quad)$

8. Factor each expression:

a. $2x^2 - 16x$ b. $3n^2 + 9n$ c. $4a^2 + 4$

d. $7u^2 + 9u$ e. $2a^2 - 16$ f. $4x^2 + 8y$

g. $5b^2 - 10b$ h. $7w^2 + 21w$ i. $7x^2 + 8x$

j. $7x + 9y$ k. $12x^2 - 12x$ l. $17Q + 17R$

m. $15g - 45h$ n. $ab + ac$ o. $xy - yz$

p. $-2x + 8$ q. $-10n - 15m$ r. $6e - 19f$

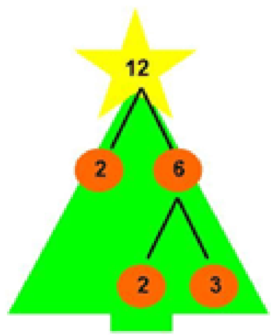
Review Problems

9. Factor each expression:

a. $3x - 12$ b. $9x + 9$ c. $7y^2 - 14y$

d. $2n^2 - 10n$ e. $10w^2 + 45w$ f. $8x + 13$

g. $-x - 4$ h. $14n^2 - 21n + 35$ i. $20n^2 - 20n$



Solutions

1. a. Infinitely many b. 2 c. 2 d. 25

e. NO — Whatever that number actually is, it's certainly divisible by both 123 and 56,893. That's too many factors to be prime.

2. a. $306 = 2 \times 3^2 \times 17$ b. 2^{10}

3. Here's what I would do. First, the given expression, $20x + 30y$, consists of two terms, and the result, $10(2x + 3y)$, consists of one term. Since factoring is the process of converting two or more terms into a single term, so far so good. Moreover, if I take my answer, $10(2x + 3y)$, and distribute to remove the parentheses, I will get $20x + 30y$, the original problem. I hope your buddy is now convinced.

4. While it may be true that the original expression consists of two terms, and her answer consists of one term, there's still one big problem. Ask her to take her answer, $6(w + 3z)$, and distribute it to remove the parentheses. She will get $6w + 18z$, which is not equal to the original problem. Therefore, her factorization can't possibly be right.

5. a. $x + z$ b. $P - Q$ c. $x - 4$ d. $2y - 3t$
e. $u + 1$ f. $n - 4$ g. $a - b$ h. $c + d$
i. $x + 2y - 4z$ j. $w - u + z$ k. $2x - 3$ l. $2a^2 + 3a - 4$

6. a. $3(P + Q)$ b. $9(n - 3)$ c. $n(c + d)$
d. $x(w - y)$ e. $7(t - 1)$ f. $x(1 + y)$
g. $-2(4L - 5)$ h. $-b(a + c)$ i. $-(u + 5)$
j. $-(z + x - 10)$ k. $2(x + y + z)$ l. $5(a - 2b + 3c)$

7. a. $a + 2b$ b. $3u - 1$ c. $Q - 3R$ d. $3x + 2$
e. $y - 2$ f. $2a + 3b$ g. $t + 4$ h. $3w - 4z$
i. $5a - 4$ j. $y + 1$

8. a. $2x(x - 8)$ b. $3n(n + 3)$ c. $4(a^2 + 1)$
d. $u(7u + 9)$ e. $2(a^2 - 8)$ f. $4(x^2 + 2y)$
g. $5b(b - 2)$ h. $7w(w + 3)$ i. $x(7x + 8)$
j. Not factorable k. $12x(x - 1)$ l. $17(Q + R)$
m. $15(g - 3h)$ n. $a(b + c)$ o. $y(x - z)$
p. $-2(x - 4)$ q. $-5(2n + 3m)$ r. Not factorable

9. a. $3(x - 4)$ b. $9(x + 1)$ c. $7y(y - 2)$
d. $2n(n - 5)$ e. $5w(2w + 9)$ f. Not factorable
g. $-(x + 4)$ h. $7(2n^2 - 3n + 5)$ i. $20n(n - 1)$

❑ *TO ∞ AND BEYOND*

1. Factor: $\pi^4x^2 + \pi^3x^3 - \pi^2x^4$

2. The price of a can of root beer is more than \$0.20. How many cans of root beer could you buy for exactly \$4.37?

"<u>Give</u> a man a fish

and you feed him for a day.

<u>Teach</u> a man to fish

and you feed him for a lifetime."

Chinese Proverb